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# FUNDAMENTAL SOLUTIONS OF THE PLANE AND THREE-DIMENSIONAL BIPOTENTIAL OPERATORS DEFINED ON VARIOUS TYPES OF RIEMANN SURFACES AND SPACES

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**Abstract**—Starting from well-known fundamental solutions of the potential operators defined on Riemann surfaces and spaces the fundamental solutions of the bipotential operators are established. They can be used as the kernels of boundary integral equations (e.g. for plate bending problems) and should be particularly suited if the domains have slits or cracks or overlapping parts.

## 1. INTRODUCTION

Integral equation formulations for boundary value problems usually contain fundamental solutions (Ortner, 1980) of the corresponding differential operators as the kernels. Mostly fundamental solutions defined on the ordinary space are used [see e.g. Cruse and Rizzo (1975) and Heise (1978)] because of their simple structure (e.g. $(\ln R)/(2\pi)$ in the case of the plane potential problem and Kelvin's solution in the case of the elastostatic problem). The treatment of boundary integral equations of this type is problematic when parts of the boundary of the considered domain touch or almost touch each other. Examples for such domains are elastic bodies having slits or cracks and plane bodies parts of which overlap each other. The reason for the difficulties is that boundary values can be prescribed on both parts of the surface which face one another, independently of each other, whereas the solution of the integral equation on one part is dependent on that on the other part. Collocation of the integral equation at two points on opposite parts of the surface facing each other yields two algebraic equations the coefficients of which coincide, i.e. the two equations are generally contradictory.

A possibility to overcome the difficulties consists in using as the kernels of the integral equations instead of fundamental solutions defined on the ordinary plane or space fundamental solutions defined on Riemann surfaces or spaces.

The best way to explain a Riemann surface is to describe how it can be produced. For this purpose we take a sheet of paper, mark two points on it and draw a simple curve between the points. Then we make $n-1$ photocopies so that we have $n$ identical sheets at our disposal. The sheets are cut along the curves and we denote the opposite borders of the cuts by $+$ and $-$. Next, we join the $+$ border of the first sheet to the $-$ border of the second sheet and the $+$ border of the second sheet to the $-$ border of the third sheet, etc. There remain the $-$ border of the first sheet and the $+$ border of the $n$th sheet. They shall formally be connected with each other which, however, surpasses our imagination. Once having joined the individual sheets one cannot find out where precisely they have been connected, i.e. the particular shape of the cuts is insignificant. Only the positions of their beginning and end points are characteristic for the Riemann surface. The two points represent branch points of order $n-1$.

A Riemann space with $n$ "sheets" is the three-dimensional generalization of a Riemann surface with $n$ sheets (Heise, 1991a).

Using fundamental solutions defined on Riemann spaces as the kernels of integral equations the above mentioned difficulties can be avoided. For this purpose the domain in question is "embedded" in the Riemann space in such a manner that the parts of the surface of the domain facing each other are situated on different sheets of the Riemann space.

Originally opposite points are no longer adjacent to each other. Collocation of the integral equation at two such points leads to two independent algebraic equations.

In preceding papers (Heise, 1990, 1991a, b, 1992a, b, to appear a, b) the author has established fundamental solutions of the plane and three-dimensional Laplace and Navier operators (differential operators for the potential and the elasticity problem) defined on various kinds of Riemann surfaces and spaces.

In Heise (1990) a first attempt has been made to find the fundamental solution of the bipotential operator defined on a Riemann surface with one branch point and an infinite number of sheets. The result is unsatisfactory. Afterwards this solution has been established in Heise (1992b) by relatively complicated calculations with the aid of the method of separation of the variables, whereas an attempt to solve an analogous three-dimensional problem, viz. to obtain the fundamental solution of the bipotential operator defined on a Riemann space with an infinitely long straight branch line and an infinite number of sheets did not lead to a final result.

Inspection of the classical fundamental solutions of the potential and bipotential operators defined on the ordinary plane and space (see Sections 3, 4) suggests a simple method for establishing the solutions of the bipotential operator for quite general domains of definition if the solutions of the potential operator for these domains are well known. In Section 2 of the paper at hand it is shown that multiplication of a fundamental solution of the potential operator by the square of the distance between the field point and the source point yields, apart from a constant factor, the fundamental solution of the bipotential operator.

In Sections 5–9 the fundamental solutions $F$ of the potential operator, defined on various types of Riemann surfaces and spaces with finite and infinite numbers of sheets, which have been established in Heise (1990, 1991a, b, 1992a, b, to appear a, b) and Sommerfeld (1897), are listed and the fundamental solutions $G$ of the bipotential operator are deduced from them. (Section 5: Riemann surface with one branch point. Section 6: Riemann surface with two branch points. Section 7: Riemann space with one infinitely long straight branch line. Section 8: Riemann space with a circular branch line. Section 9: Riemann space with two infinitely long, parallel straight branch lines.)

In the following a short survey is given of various methods which have been used in Heise (1990, 1991a, b, 1992a, b, to appear a, b) and Sommerfeld (1897) for establishing fundamental solutions of the Laplace operator defined on Riemann surfaces and spaces. The methods are not at all equally suited for the different types of surfaces and spaces.

A possibility exists of transforming the potential operator intuitively into an operator, the fundamental solution of which is well known (Heise, 1990). In Heise (1992b) the problem is reduced by separation of the independent variables to finding the fundamental solution of an ordinary differential operator. Sommerfeld (1897) has proposed an ingenious method based on Cauchy's integral formula [see Heise (1991a, to appear a, b)]. Conformal mapping represents the simplest method which, however, is only applicable for plane problems. We consider it a bit closer. It starts from the fundamental solution

$$\frac{1}{4\pi}\ln|z-\bar{z}|^2 \tag{1}$$

of the Laplace operator defined on the complex plane. $z = x+iy$ indicates the field point and $\bar{z} = \bar{x}+i\bar{y}$ the source point. The transformation $z \to z^{1/n}$ maps the plane onto a Riemann surface with one branch point and $n$ sheets. The corresponding fundamental solution reads as [see Heise (1992a); Section 5]:

$$F = \frac{1}{4\pi}\ln(n^2|z^{1/n}-\bar{z}^{1/n}|^2). \tag{2}$$

Replacing the Cartesian co-ordinates $x, y$ by polar co-ordinates $r, \varphi$ and neglecting a superfluous summand we obtain the final result (26). The fundamental solution (28) defined on a Riemann suface with two branch points and $n$ sheets is obtained similarly using the transformation [see Heise (1991b); Section 5]:

$$z \rightarrow \frac{a^{1/n}}{2} \frac{(2z+a)^{1/n}+(2z-a)^{1/n}}{(2z+a)^{1/n}-(2z-a)^{1/n}}. \tag{3}$$

The fundamental solutions of the plane bipotential operator obtained in this paper can be used as the kernels of integral equations for plate bending problems and should be particularly suited if the plate has slits or cracks or overlapping parts (helical elastic springs with rectangular cross-sections). The three-dimensional bipotential operator occurs in certain formulations of elastostatic problems (Galerkin vector, stress function tensor) (Lurje, 1963).

Various types of boundary integral equations (equations of the first and second kind, direct and indirect formulations) with classical fundamental solutions as kernels represent frequently used tools for solving engineering problems. Pendants of each of these equations can be easily established. One has only to replace the classical fundamental solution defined on the ordinary space by a fundamental solution defined on a Riemann space.

It is possible to treat the proposed integral equations numerically in quite the same manner as the classical equations. Numerical results for plane potential problems confirm the expectation that the magnitude of the error does not depend considerably on whether the kernel is defined on the ordinary space or on a Riemann space the number of sheets of which is not excessively large (Heise, 1991a, b, 1992a, b, to appear a).

In Section 10 it is shown by a simple numerical example that the recommended method also works in the case of bipotential problems.

## 2. A METHOD FOR ESTABLISHING FUNDAMENTAL SOLUTIONS OF THE BIPOTENTIAL OPERATOR USING FUNDAMENTAL SOLUTIONS OF THE POTENTIAL OPERATOR

Comparing the classical fundamental solutions [see (22), (23), (24), (25)] of the potential and bipotential operators defined on the ordinary two- and three-dimensional spaces we observe that the solutions of the equation

$$\boxed{\Delta\Delta G = \delta} \tag{4}$$

can be written as

$$\boxed{G = B \cdot R^2 F,} \tag{5}$$

where $F$ fulfils the equation

$$\boxed{\Delta F = \delta,} \tag{6}$$

and where $B$ is a constant factor, the value of which depends on the dimension of the problem, where $R$ is the distance between the field point (radius vector $x_t$) and the source point (radius vector $\bar{x}_t$)

$$R^2 = (x_t - \bar{x}_t)^2, \tag{7}$$

and where $\delta = \delta(x_t - \bar{x}_t)$ is Dirac's function. In the following we prove that eqns (4)–(6) also hold true when the domain of definition is a more general space, i.e. we prove that we obtain for example a fundamental solution $G$ of the bipotential operator defined on a

Riemann surface or on a Riemann space by insertion into (5) of a fundamental solution $F$ of the potential operator defined on a Riemann surface or space. We find

$$\partial_i R^2 = 2(x_i - \bar{x}_i), \qquad \partial_j \partial_i R^2 = 2\delta_{ij}, \tag{8),(9}$$

$$\Delta R^2 = \partial_j \partial_j R^2 = 2\delta_{jj}, \qquad \partial_i \Delta R^2 = \Delta \partial_i R^2 = 0, \tag{10),(11}$$

$$\Delta G = B\Delta(R^2 F) = B[R^2\delta + 2\delta_{jj}F + 4(x_i - \bar{x}_i)\partial_i F], \tag{12}$$

$$\Delta\Delta G = B[4(2+\delta_{jj})\delta + R^2\Delta\delta + 8(x_j - \bar{x}_j)\partial_j \delta], \tag{13}$$

where $\delta_{ij}$ is the identity tensor. We consider integrals of the third and second summand of (13) extended over an arbitrary domain $A$ of the plane or space which includes the source point $\bar{x}_i$ and which is bounded by the curve or surface $\Gamma$:

$$\begin{aligned}\int_{(A)} (x_j - \bar{x}_j)\partial_j \delta \, \mathrm{d}A &= \int_{(A)} \partial_j[(x_j - \bar{x}_j)\delta] \, \mathrm{d}A - \int_{(A)} \delta \partial_j (x_j - \bar{x}_j) \, \mathrm{d}A \\ &= \int_{\Gamma} [n_j (x_j - \bar{x}_j)\delta] \, \mathrm{d}\Gamma - \int_{(A)} \delta_{jj} \delta \, \mathrm{d}A \\ &= -\delta_{jj} \int_{(A)} \delta \, \mathrm{d}A, \end{aligned} \tag{14}$$

$$\begin{aligned}\int_{(A)} (x_i - \bar{x}_i)^2 \Delta\delta \, \mathrm{d}A &= \int_{(A)} (x_i - \bar{x}_i)^2 (\partial_j \partial_j \delta) \, \mathrm{d}A \\ &= \int_{(A)} \partial_j [(x_i - \bar{x}_i)^2 \partial_j \delta] \, \mathrm{d}A - \int_{(A)} [\partial_j (x_i - \bar{x}_i)^2][\partial_j \delta] \, \mathrm{d}A \\ &= \int_{(\Gamma)} n_j [(x_i - \bar{x}_i)^2 \partial_j \delta] \, \mathrm{d}\Gamma - 2\int_{(A)} (x_j - \bar{x}_j)(\partial_j \delta) \, \mathrm{d}A \\ &= 0 + 2\delta_{jj} \int_{(A)} \delta \, \mathrm{d}A. \end{aligned} \tag{15}$$

The integrals over the boundary $\Gamma$ of the domain $A$ occurring in (14), (15) vanish since the factors $\delta = \delta(x_i - \bar{x}_i)$ and $\partial_j \delta$ of the integrands are equal to zero for $x_i \neq \bar{x}_i$. Comparing the first and the last integrals of (14), (15) we conclude:

$$(x_j - \bar{x}_j)\partial_j \delta = -\delta_{jj}\delta, \tag{16}$$

$$R^2 \Delta\delta = (x_j - \bar{x}_j)^2 \Delta\delta = 2\delta_{jj}\delta. \tag{17}$$

Inserting (16), (17) into (13) we obtain:

$$\Delta\Delta G = B[8 - 2\delta_{jj}]\delta, \tag{18}$$

$$B = \frac{1}{8 - 2\delta_{jj}}. \tag{19}$$

| | | |
|---|---|---|
| $B = \frac{1}{2}$ | three-dimensional problem, | (20) |
| $B = \frac{1}{4}$ | two-dimensional problem. | (21) |

We deal only with plane and three-dimensional problems. Nevertheless it is interesting to observe that in the four-dimensional case (5) is not valid since (19) yields $B = \infty$.

## 3. SOLUTIONS FOR THE ORDINARY PLANE

$$F = \frac{1}{2\pi} \ln R, \qquad G = \frac{R^2}{4} F. \tag{22), (23}$$

## 4. SOLUTIONS FOR THE ORDINARY SPACE

$$F = -\frac{1}{4\pi} \frac{1}{R}, \qquad G = \frac{R^2}{2} F. \tag{24), (25}$$

## 5. SOLUTIONS FOR A RIEMANN SURFACE WITH ONE BRANCH POINT AND $n$ SHEETS

[See Heise (1992a): eqns (5.3), (5.8), (5.9), (5.16) and Heise (1992b): eqns (19), (36)]

$$F = \frac{1}{4\pi} \ln \left\{ r^{2/n} + \bar{r}^{2/n} - 2(r\bar{r})^{1/n} \cos \frac{\varphi - \bar{\varphi}}{n} \right\}, \tag{26}$$

$$F = \frac{1}{4\pi} \ln \left\{ \left( \ln \frac{r}{\bar{r}} \right)^2 + (\varphi - \bar{\varphi})^2 \right\}, \quad \text{for } n = \infty, \tag{26a}$$

$$G = \frac{R^2}{4} \cdot F = \tfrac{1}{4}(r^2 + \bar{r}^2 - 2r\bar{r} \cos(\varphi - \bar{\varphi}))F, \tag{27}$$

where $r, \varphi, \bar{r}, \bar{\varphi}$ are polar co-ordinates of the field point and the source point.

## 6. SOLUTIONS FOR A RIEMANN SURFACE WITH TWO BRANCH POINTS AND $n$ SHEETS

[See Heise (1991b): eqns (5.6), (5.9), (4.17)]

$$F = \frac{1}{4\pi} \ln \frac{a^{2/n}}{2} \frac{\cosh \frac{r - \bar{r}}{n} - \cos \frac{\varphi - \bar{\varphi}}{n}}{\left( \cosh \frac{\bar{r}}{n} - \cos \frac{\bar{\varphi}}{n} \right) \left( \cosh \frac{r}{n} - \cos \frac{\varphi}{n} \right)}, \tag{28}$$

$$F = \frac{1}{4\pi} \ln \frac{(r - \bar{r})^2 + (\varphi - \bar{\varphi})^2}{(\bar{r}^2 + \bar{\varphi}^2)(r^2 + \varphi^2)} \quad \text{for } n = \infty, \tag{28a}$$

$$G = \frac{R^2}{4} \cdot F = \frac{1}{4} \frac{a^2}{2} \frac{\cosh(r - \bar{r}) - \cos(\varphi - \bar{\varphi})}{(\cosh \bar{r} - \cos \bar{\varphi})(\cosh r - \cos \varphi)} F, \tag{29}$$

where $r, \varphi, \bar{r}, \bar{\varphi}$ are bipolar co-ordinates of the field point and of the source point [for details see Heise (1991b)]. $a$ is the distance between the two branch points.

## 7. SOLUTIONS FOR A RIEMANN SPACE WITH AN INFINITELY LONG STRAIGHT BRANCH LINE AND *n* SHEETS

[See Heise (1991a): eqns (2.33), (2.34), (2.1), (2.2) and Heise (1992b): eqn (61)]

$$F=\frac{1}{4\pi^2}\frac{1}{n}\frac{1}{\sqrt{2r\bar{r}}}\int_{\alpha_1}^{\infty}\frac{d\kappa}{\sqrt{\cosh\kappa-\cosh\alpha_1}}\frac{\sinh\frac{\kappa}{n}}{\cos\frac{\varphi-\bar{\varphi}}{n}-\cosh\frac{\kappa}{n}},\tag{30}$$

$$F=-\frac{1}{2\pi^2}\frac{1}{\sqrt{2r\bar{r}}}\int_{\alpha_1}^{\infty}\frac{d\kappa}{\sqrt{\cosh\kappa-\cosh\alpha_1}}\frac{\kappa}{\kappa^2+(\varphi-\bar{\varphi})^2},\quad \text{for } n=\infty,\tag{30a}$$

$$G=\frac{R^2}{2}F=\tfrac{1}{2}\{r^2+\bar{r}^2+(z-\bar{z})^2-2r\bar{r}\cos(\varphi-\bar{\varphi})\}\cdot F,\tag{31}$$

$$\cosh\alpha_1=\{r^2+\bar{r}^2+(z-\bar{z})^2\}/(2r\bar{r}),\tag{32}$$

where $r, \varphi, z, \bar{r}, \bar{\varphi}, \bar{z}$ are cylinder co-ordinates of the field point and of the source point. The integration in (30) can be carried out analytically for $n=2$ [see Heise (1991a)].

## 8. SOLUTIONS FOR A RIEMANN SPACE WITH A CIRCULAR BRANCH LINE AND *n* SHEETS

[See Heise (to appear b): eqns (4.12), (3.9), (4.2)]

$$F=\frac{1}{4\pi^2}\frac{1}{n}\frac{\sqrt{2}}{a}\sqrt{(\cosh r-\cos\varphi)(\cosh\bar{r}-\cos\bar{\varphi})}\cdot\int_{\alpha_1}^{\infty}\frac{d\kappa}{\sqrt{\cosh\kappa-\cosh\alpha_1}}\frac{\sinh\frac{\kappa}{n}}{\cos\frac{\varphi-\bar{\varphi}}{n}-\cosh\frac{\kappa}{n}},\tag{33}$$

$$F=-\frac{1}{2\pi^2}\frac{\sqrt{2}}{a}\sqrt{(\cosh r-\cos\varphi)(\cosh\bar{r}-\cos\bar{\varphi})}\cdot\int_{\alpha_1}^{\infty}\frac{d\kappa}{\sqrt{\cosh\kappa-\cosh\alpha_1}}\frac{\kappa}{\kappa^2+(\varphi-\bar{\varphi})^2},\quad \text{for } n=\infty,\tag{33a}$$

$$G=\frac{R^2}{2}F=\frac{1}{2}\frac{a^2}{2}\frac{\cosh(r-\bar{r})+(1-\cos(\omega-\bar{\omega}))\sinh r\sinh\bar{r}-\cos(\varphi-\bar{\varphi})}{(\cosh r-\cos\varphi)(\cosh\bar{r}-\cos\bar{\varphi})}\cdot F,\tag{34}$$

$$\cosh\alpha_1=\cosh(r-\bar{r})+(1-\cos(\omega-\bar{\omega}))\sinh r\sinh\bar{r},\tag{35}$$

where $r, \varphi, \omega, \bar{r}, \bar{\varphi}, \bar{\omega}$ are toroidal co-ordinates of the field point and of the source point [for details see Heise (to appear b)]. $a/2$ is the radius of the circle.

The integration in (33) can be carried out analytically for $n=2$ [see Heise (to appear b)].

## 9. SOLUTIONS FOR A RIEMANN SPACE WITH TWO INFINITELY LONG, PARALLEL STRAIGHT BRANCH LINES AND $n$ SHEETS

[See Heise (to appear a): eqns (2.64), (2.6)]

$$F = -\frac{1}{\sqrt{2}}\frac{1}{4\pi^2}\frac{1}{n}\sqrt{\frac{2}{a^2}(\cosh r-\cos\varphi)(\cosh\bar{r}-\cos\bar{\varphi})}$$

$$\cdot\int_0^\infty \frac{\mathrm{d}w}{\sqrt{\sinh\frac{w}{2}}}\,\frac{1}{\sqrt{C\sinh\frac{w}{2}+\sqrt{C^2-D^2}\cosh\frac{w}{2}}}\,\frac{\sinh\frac{w+\alpha_1}{n}}{\cosh\frac{w+\alpha_1}{n}-\cos\frac{\psi-\bar{\varphi}}{n}}. \tag{36}$$

[For $\alpha_1 = \alpha_1(r,\bar{r},\varphi,A)$, $\psi = \psi(\varphi,A)$, $C = C(r,\bar{r},A)$, $D = D(\varphi,A,\psi)$, $A = A(r,\varphi,z,\bar{z})$ see Heise (to appear a): eqns (2.52), (2.38), (2.39), (2.20), (2.50), (2.19).]

In Section 2.5 of Heise (to appear a) the solution $F$ defined on a Riemann space with an infinite number of sheets is given.

The solution of the bipotential operator reads as:

$$G = \frac{R^2}{2}\cdot F = \frac{1}{2}\left\{\frac{a^2}{2}\,\frac{\cosh(r-\bar{r})-\cos(\varphi-\bar{\varphi})}{(\cosh\bar{r}-\cos\bar{\varphi})(\cosh r-\cos\varphi)}+(z-\bar{z})^2\right\}F, \tag{37}$$

where $r,\varphi,\bar{r},\bar{\varphi}$ are bipolar co-ordinates of the field point and the source point [for details concerning bipolar co-ordinates see Heise (1991b)]. The $z$- and $\bar{z}$-axes are perpendicular to the $r$–$\varphi$-plane and $\bar{r}$–$\bar{\varphi}$-plane, respectively.

For $n = 2$ the integration in (36) can be carried out analytically [see Heise (to appear a)].

## 10. NUMERICAL RESULTS FOR A SIMPLE EXAMPLE

The aim of this paper is to give a list of fundamental solutions of the bipotential operator defined on various Riemann surfaces and spaces. The solutions are suited as the kernels of integral equations for boundary value problems governed by the bipotential equation. By applying these equations certain difficulties can be circumvented which are inherent in classical boundary integral equations if parts of the boundary of the domain touch each other (e.g. domains with slits or cracks) or if plane domains have overlapping parts.

The details of the establishment of the integral equations and their numerical treatment represent separately treatable subjects. As a matter of fact it is only necessary to exchange the kernels of the classical equations by the fundamental solutions defined on Riemann surfaces and spaces. Numerical results in Heise (1991a, b, 1992a, b, to appear a) demonstrate that the proposed concept is feasible in the case of potential problems. Dealing with bipotential problems no principally different aspects have to be considered. Only the necessary effort is larger since two conditions have to be fulfilled instead of one on the boundary.

In this section we will show by a very simple example that the method also works in the case of bipotential problems.

There are various integral equation formulations to choose from. We consider the simplest possible formulation, a system of two equations:

$$\oint G(s,\bar{s})b(\bar{s})\,\mathrm{d}\bar{s}+\oint\frac{\partial G}{\partial\bar{n}}(s,\bar{s})h(\bar{s})\,\mathrm{d}\bar{s} = \Phi(s), \tag{38a}$$

$$\oint\frac{\partial G}{\partial n}(s,\bar{s})b(\bar{s})\,\mathrm{d}\bar{s}+\oint\frac{\partial^2 G}{\partial n\,\partial\bar{n}}(s,\bar{s})h(\bar{s})\,\mathrm{d}\bar{s} = \frac{\partial\Phi}{\partial n}(s), \tag{38b}$$

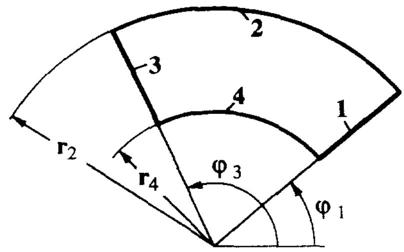


Fig. 1. Problem geometry. Sector of a circular ring bounded by the straight lines 1, 3 and the arcs 2, 4.

in which a layer $b(s)$ of sources and a layer $h(s)$ of dipoles represent the sought after functions. The kernel $G$ is the fundamental solution (27) defined on a Riemann surface with one branch point and $n$ sheets. For brevity we have written $G(s, \bar{s})$ instead of $G(r(s), \varphi(s), \bar{r}(\bar{s}), \bar{\varphi}(\bar{s}))$. $\bar{s}$ is the arc length of the source point and $s$ is the arc length of the field point. By $\partial/\partial\bar{n}$ we denote the normal derivative at the source point and by $\partial/\partial n$ the normal derivative at the field point. $\Phi(s)$ and $(\partial\Phi/\partial n)(s)$ are given boundary values. Equation (38) represents for example an integral equation formulation for the determination of the deflection $\Phi$ of a bent thin elastic plate the boundary of which is clamped in a non-plane flange. The two auxiliary functions $b(\bar{s})$ and $h(\bar{s})$ can be interpreted as layers of forces and of bending moments respectively.

Since three of the four kernels of (38) do not diverge at $s = \bar{s}$ but vanish, discretization of the equations yields relatively ill-conditioned algebraic systems. Therefore, equations which contain higher order derivatives of the fundamental solutions as the kernels should be better suited for practical purposes. However, this is not a consequence of using fundamental solutions defined on Riemann surfaces but occurs exactly in the same manner when dealing with classical fundamental solutions!

We consider a simple domain, namely a sector of a circular ring (see Figs 1, 2) the aperture angle of which exceeds $2\pi$, i.e. parts of the domain overlap each other.

The system of integral equations (38) can be written as

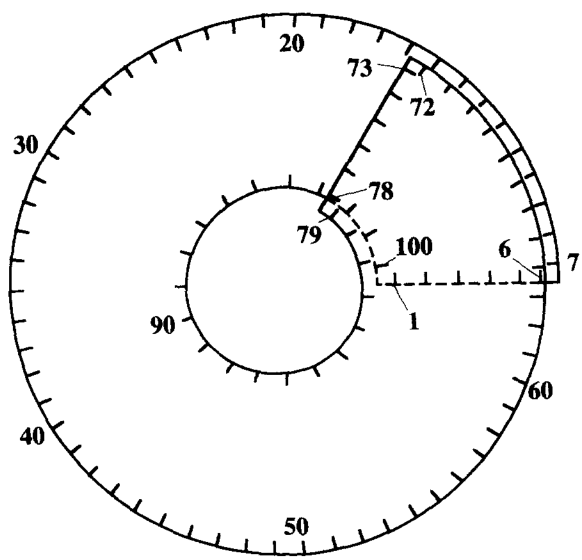


Fig. 2. Domain having the shape of a circular ring with $r_2 = 3$, $r_4 = 1$, $\varphi_1 = 0$, $\varphi_2 = 7.33333$. The aperture angle exceeds $2\pi$, i.e. two parts of the domain overlap each other. The circles are drawn on purpose in a slightly distorted manner. The boundary of the domain is divided into 100 elements. The central points of these elements are marked.

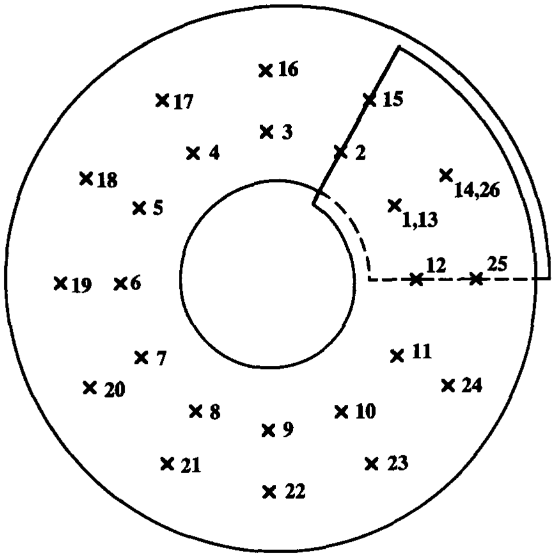


Fig. 3. Twenty-six points in the interior of the domain are marked. The points 2, 15, 12, 25 do not lie on the boundary. The points 1 and 13 (14 and 26) are not identical; point 13 (26) is situated "above" point 1 (14).

$$\sum_{j=1}^{4}\int_{(S_j)} G_{ij} b_j \,\mathrm{d}\bar{s} + \sum_{j=1}^{4}\int_{(S_j)} H_{ij} h_j \,\mathrm{d}\bar{s} = \Phi_i, \quad i = 1,2,3,4, \tag{39a}$$

$$\sum_{j=1}^{4}\int_{(S_j)} K_{ij} b_j \,\mathrm{d}\bar{s} + \sum_{j=1}^{4}\int_{(S_j)} L_{ij} h_j \,\mathrm{d}\bar{s} = \left(\frac{\partial \Phi}{\partial n}\right)_i, \quad i = 1,2,3,4, \tag{39b}$$

where the subscripts $i$ and $j$ indicate the four parts of the boundary (see Fig. 1). For example $S_2$ is the circular arc with radius $r_2$, $b_2$ is the sought-after layer of forces on $S_2$ and $\Phi_2$ is the prescribed boundary deflection on $S_2$. The kernels are given by:

$$\begin{aligned}
G_{11} &= G(r,\varphi_1,\bar{r},\varphi_1), & G_{21} &= G(r_2,\varphi,\bar{r},\varphi_1),\\
G_{12} &= G(r,\varphi_1,r_2,\bar{\varphi}), & G_{22} &= G(r_2,\varphi,r_2,\bar{\varphi}),\\
G_{13} &= G(r,\varphi_1,\bar{r},\varphi_3), & G_{23} &= G(r_2,\varphi,\bar{r},\varphi_3),\\
G_{14} &= G(r,\varphi_1,r_4,\bar{\varphi}), & G_{24} &= G(r_2,\varphi,r_4,\bar{\varphi}),\\
G_{31} &= G(r,\varphi_3,\bar{r},\varphi_1), & G_{41} &= G(r_4,\varphi,\bar{r},\varphi_1),\\
G_{32} &= G(r,\varphi_3,r_2,\bar{\varphi}), & G_{42} &= G(r_4,\varphi,r_2,\bar{\varphi}),\\
G_{33} &= G(r,\varphi_3,\bar{r},\varphi_3), & G_{43} &= G(r_4,\varphi,\bar{r},\varphi_3),\\
G_{34} &= G(r,\varphi_3,r_4,\bar{\varphi}), & G_{44} &= G(r_4,\varphi,r_4,\bar{\varphi}),
\end{aligned} \tag{40}$$

$$\begin{aligned}
H_{11} &= -\frac{1}{\bar{r}}\frac{\partial G}{\partial \bar{\varphi}}(r,\varphi_1,\bar{r},\varphi_1), & H_{21} &= -\frac{1}{\bar{r}}\frac{\partial G}{\partial \bar{\varphi}}(r_2,\varphi,\bar{r},\varphi_1),\\
H_{12} &= \frac{\partial G}{\partial \bar{r}}(r,\varphi_1,r_2,\bar{\varphi}), & H_{22} &= \frac{\partial G}{\partial \bar{r}}(r_2,\varphi,r_2,\bar{\varphi}),\\
H_{13} &= \frac{1}{\bar{r}}\frac{\partial G}{\partial \bar{\varphi}}(r,\varphi_1,\bar{r},\varphi_3), & H_{23} &= \frac{1}{\bar{r}}\frac{\partial G}{\partial \bar{\varphi}}(r_2,\varphi,\bar{r},\varphi_3),\\
H_{14} &= -\frac{\partial G}{\partial \bar{r}}(r,\varphi_1,r_4,\bar{\varphi}), & H_{24} &= -\frac{\partial G}{\partial \bar{r}}(r_2,\varphi,r_4,\bar{\varphi}),
\end{aligned}$$

$$H_{31} = -\frac{1}{\bar{r}}\frac{\partial G}{\partial\bar{\varphi}}(r,\varphi_3,\bar{r},\varphi_1),\quad H_{41} = -\frac{1}{\bar{r}}\frac{\partial G}{\partial\bar{\varphi}}(r_4,\varphi,\bar{r},\varphi_1),$$

$$H_{32} = \frac{\partial G}{\partial\bar{r}}(r,\varphi_3,r_2,\bar{\varphi}),\quad H_{42} = \frac{\partial G}{\partial\bar{r}}(r_4,\varphi,r_2,\bar{\varphi}),$$

$$H_{33} = \frac{1}{\bar{r}}\frac{\partial G}{\partial\bar{\varphi}}(r,\varphi_3,\bar{r},\varphi_3),\quad H_{43} = \frac{1}{\bar{r}}\frac{\partial G}{\partial\bar{\varphi}}(r_4,\varphi,\bar{r},\varphi_3),$$

$$H_{34} = -\frac{\partial G}{\partial\bar{r}}(r,\varphi_3,r_4,\bar{\varphi}),\quad H_{44} = -\frac{\partial G}{\partial\bar{r}}(r_4,\varphi,r_4,\bar{\varphi}), \tag{41}$$

$$K_{11} = -\frac{1}{r}\frac{\partial G}{\partial\varphi}(r,\varphi_1,\bar{r},\varphi_1),\quad K_{21} = \frac{\partial G}{\partial r}(r_2,\varphi,\bar{r},\varphi_1),$$

$$K_{12} = -\frac{1}{r}\frac{\partial G}{\partial\varphi}(r,\varphi_1,r_2,\bar{\varphi}),\quad K_{22} = \frac{\partial G}{\partial r}(r_2,\varphi,r_2,\bar{\varphi}),$$

$$K_{13} = -\frac{1}{r}\frac{\partial G}{\partial\varphi}(r,\varphi_1,\bar{r},\varphi_3),\quad K_{23} = \frac{\partial G}{\partial r}(r_2,\varphi,\bar{r},\varphi_3),$$

$$K_{14} = -\frac{1}{r}\frac{\partial G}{\partial\varphi}(r,\varphi_1,r_4,\bar{\varphi}),\quad K_{24} = \frac{\partial G}{\partial r}(r_2,\varphi,r_4,\bar{\varphi}),$$

$$K_{31} = \frac{1}{r}\frac{\partial G}{\partial\varphi}(r,\varphi_3,\bar{r},\varphi_1),\quad K_{41} = -\frac{\partial G}{\partial r}(r_4,\varphi,\bar{r},\varphi_1),$$

$$K_{32} = \frac{1}{r}\frac{\partial G}{\partial\varphi}(r,\varphi_3,r_2,\bar{\varphi}),\quad K_{42} = -\frac{\partial G}{\partial r}(r_4,\varphi,r_2,\bar{\varphi}),$$

$$K_{33} = \frac{1}{r}\frac{\partial G}{\partial\varphi}(r,\varphi_3,\bar{r},\varphi_3),\quad K_{43} = -\frac{\partial G}{\partial r}(r_4,\varphi,\bar{r},\varphi_3),$$

$$K_{34} = \frac{1}{r}\frac{\partial G}{\partial\varphi}(r,\varphi_3,r_4,\bar{\varphi}),\quad K_{44} = -\frac{\partial G}{\partial r}(r_4,\varphi,r_4,\bar{\varphi}), \tag{42}$$

$$L_{11} = \frac{1}{r\bar{r}}\frac{\partial^2 G}{\partial\varphi\,\partial\bar{\varphi}}(r,\varphi_1,\bar{r},\varphi_1),\quad L_{21} = -\frac{1}{\bar{r}}\frac{\partial^2 G}{\partial r\,\partial\bar{\varphi}}(r_2,\varphi,\bar{r},\varphi_1),$$

$$L_{12} = -\frac{1}{r}\frac{\partial^2 G}{\partial\varphi\,\partial\bar{r}}(r,\varphi_1,r_2,\bar{\varphi}),\quad L_{22} = \frac{\partial^2 G}{\partial r\,\partial\bar{r}}(r_2,\varphi,r_2,\bar{\varphi}),$$

$$L_{13} = -\frac{1}{r\bar{r}}\frac{\partial^2 G}{\partial\varphi\,\partial\bar{\varphi}}(r,\varphi_1,\bar{r},\bar{\varphi}_3),\quad L_{23} = \frac{1}{\bar{r}}\frac{\partial^2 G}{\partial r\,\partial\bar{\varphi}}(r_2,\varphi,\bar{r},\varphi_3),$$

$$L_{14} = \frac{1}{r}\frac{\partial^2 G}{\partial\varphi\,\partial\bar{r}}(r,\varphi_1,r_4\,\bar{\varphi}),\quad L_{24} = -\frac{\partial^2 G}{\partial r\,\partial\bar{r}}(r_2,\varphi,r_4,\bar{\varphi}),$$

$$L_{31} = -\frac{1}{r\bar{r}}\frac{\partial^2 G}{\partial\varphi\,\partial\bar{\varphi}}(r,\varphi_3,\bar{r},\varphi_1),\quad L_{41} = \frac{1}{\bar{r}}\frac{\partial^2 G}{\partial r\,\partial\bar{\varphi}}(r_4,\varphi,\bar{r},\bar{\varphi}_1),$$

$$L_{32} = \frac{1}{r}\frac{\partial^2 G}{\partial\varphi\,\partial\bar{r}}(r,\varphi_3,r_2,\bar{\varphi}),\quad L_{42} = -\frac{\partial^2 G}{\partial r\,\partial\bar{r}}(r_4,\varphi,r_2,\bar{\varphi}),$$

$$L_{33} = \frac{1}{r\bar{r}}\frac{\partial^2 G}{\partial\varphi\,\partial\bar{\varphi}}(r,\varphi_3,\bar{r},\varphi_3),\quad L_{43} = -\frac{1}{\bar{r}}\frac{\partial^2 G}{\partial r\,\partial\bar{\varphi}}(r_4,\varphi,\bar{r},\varphi_3),$$

$$L_{34} = -\frac{1}{r}\frac{\partial^2 G}{\partial\varphi\,\partial\bar{r}}(r,\varphi_3,r_4,\bar{\varphi}),\quad L_{44} = \frac{\partial^2 G}{\partial r\,\partial\bar{r}}(r_4,\varphi,r_4,\bar{\varphi}), \tag{43}$$

$$G = (\ln D)B, \tag{44}$$

$$\frac{\partial G}{\partial r} = (\ln D)\frac{\partial B}{\partial r} + \frac{B}{D}\frac{\partial D}{\partial r}, \tag{45}$$

$$\frac{\partial G}{\partial \bar{r}} = (\ln D)\frac{\partial B}{\partial \bar{r}} + \frac{B}{D}\frac{\partial D}{\partial \bar{r}}, \tag{46}$$

$$\frac{1}{r}\frac{\partial G}{\partial \varphi} = \frac{1}{r}\left\{(\ln D)\frac{\partial B}{\partial \varphi} + \frac{B}{D}\frac{\partial D}{\partial \varphi}\right\}, \tag{47}$$

$$\frac{1}{\bar{r}}\frac{\partial G}{\partial \bar{\varphi}} = \frac{1}{\bar{r}}\left\{(\ln D)\frac{\partial B}{\partial \bar{\varphi}} + \frac{B}{D}\frac{\partial D}{\partial \bar{\varphi}}\right\}, \tag{48}$$

$$\frac{\partial^2 G}{\partial r\,\partial \bar{r}} = (\ln D)\frac{\partial^2 B}{\partial r\,\partial \bar{r}} + \frac{1}{D}\frac{\partial D}{\partial r}\frac{\partial B}{\partial \bar{r}} + \frac{1}{D}\frac{\partial B}{\partial r}\frac{\partial D}{\partial \bar{r}} + \frac{B}{D}\frac{\partial^2 D}{\partial r\,\partial \bar{r}} - \frac{B}{D^2}\frac{\partial D}{\partial r}\frac{\partial D}{\partial \bar{r}}, \tag{49}$$

$$\frac{1}{\bar{r}}\frac{\partial^2 G}{\partial r\,\partial \bar{\varphi}} = \frac{1}{\bar{r}}\left\{\ln D)\frac{\partial^2 B}{\partial r\,\partial \bar{\varphi}} + \frac{1}{D}\frac{\partial D}{\partial r}\frac{\partial B}{\partial \bar{\varphi}} + \frac{1}{D}\frac{\partial B}{\partial r}\frac{\partial D}{\partial \bar{\varphi}} + \frac{B}{D}\frac{\partial^2 D}{\partial r\,\partial \bar{\varphi}} - \frac{B}{D^2}\frac{\partial D}{\partial r}\frac{\partial D}{\partial \bar{\varphi}}\right\}, \tag{50}$$

$$\frac{1}{r}\frac{\partial^2 G}{\partial \varphi\,\partial \bar{r}} = \frac{1}{r}\left\{(\ln D)\frac{\partial^2 B}{\partial \varphi\,\partial \bar{r}} + \frac{1}{D}\frac{\partial D}{\partial \varphi}\frac{\partial B}{\partial \bar{r}} + \frac{1}{D}\frac{\partial B}{\partial \varphi}\frac{\partial D}{\partial \bar{r}} + \frac{B}{D}\frac{\partial^2 D}{\partial \varphi\,\partial \bar{r}} - \frac{B}{D^2}\frac{\partial D}{\partial \varphi}\frac{\partial D}{\partial \bar{r}}\right\}, \tag{51}$$

$$\frac{1}{r\bar{r}}\frac{\partial^2 G}{\partial \varphi\,\partial \bar{\varphi}} = \frac{1}{r\bar{r}}\left\{(\ln D)\frac{\partial^2 B}{\partial \varphi\,\partial \bar{\varphi}} + \frac{1}{D}\frac{\partial D}{\partial \varphi}\frac{\partial B}{\partial \bar{\varphi}} + \frac{1}{D}\frac{\partial B}{\partial \varphi}\frac{\partial D}{\partial \bar{\varphi}} + \frac{B}{D}\frac{\partial^2 D}{\partial \varphi\,\partial \bar{\varphi}} - \frac{B}{D^2}\frac{\partial D}{\partial \varphi}\frac{\partial D}{\partial \bar{\varphi}}\right\}. \tag{52}$$

[In (53)–(61) $n$ indicates the number of sheets of the Riemann surface. In the considered numerical example $n = 2$ has been chosen.]

$$D(r,\varphi,\bar{r},\bar{\varphi},n) = r^{2/n} + \bar{r}^{2/n} - 2(r\bar{r})^{1/n}\cos\frac{\varphi-\bar{\varphi}}{n}, \tag{53}$$

$$\frac{\partial D}{\partial r}(r,\varphi,\bar{r},\bar{\varphi},n) = \frac{2}{n}\frac{1}{r}\left\{r^{2/n} - (r\bar{r})^{1/n}\cos\frac{\varphi-\bar{\varphi}}{n}\right\}, \tag{54}$$

$$\frac{\partial D}{\partial \bar{r}}(r,\varphi,\bar{r},\bar{\varphi},n) = \frac{2}{n}\frac{1}{\bar{r}}\left\{\bar{r}^{2/n} - (r\bar{r})^{1/n}\cos\frac{\varphi-\bar{\varphi}}{n}\right\}, \tag{55}$$

$$\frac{1}{r}\frac{\partial D}{\partial \varphi}(r,\varphi,\bar{r},\bar{\varphi},n) = \frac{2}{n}\frac{1}{r}(r\bar{r})^{1/n}\sin\frac{\varphi-\bar{\varphi}}{n}, \tag{56}$$

$$\frac{1}{\bar{r}}\frac{\partial D}{\partial \bar{\varphi}}(r,\varphi,\bar{r},\bar{\varphi},n) = -\frac{2}{n}\frac{1}{\bar{r}}(r\bar{r})^{1/n}\sin\frac{\varphi-\bar{\varphi}}{n}, \tag{57}$$

$$\frac{\partial^2 D}{\partial r\,\partial \bar{r}}(r,\varphi,\bar{r},\bar{\varphi},n) = -\frac{2}{n^2}\frac{1}{r\bar{r}}(r\bar{r})^{1/n}\cos\frac{\varphi-\bar{\varphi}}{n}, \tag{58}$$

$$\frac{1}{r\bar{r}}\frac{\partial^2 D}{\partial\varphi\,\partial\bar{\varphi}}(r,\varphi,\bar{r},\bar{\varphi},n) = -\frac{2}{n^2}\frac{1}{r\bar{r}}(r\bar{r})^{1/n}\cos\frac{\varphi-\bar{\varphi}}{n}, \tag{59}$$

$$\frac{1}{r}\frac{\partial^2 D}{\partial\varphi\,\partial\bar{r}}(r,\varphi,\bar{r},\bar{\varphi},n) = \frac{2}{n^2}\frac{1}{r\bar{r}}(r\bar{r})^{1/n}\sin\frac{\varphi-\bar{\varphi}}{n}, \tag{60}$$

$$\frac{1}{\bar{r}}\frac{\partial^2 D}{\partial r\,\partial\bar{\varphi}}(r,\varphi,\bar{r},\bar{\varphi},n) = -\frac{2}{n^2}\frac{1}{r\bar{r}}(r\bar{r})^{1/n}\sin\frac{\varphi-\bar{\varphi}}{n}, \tag{61}$$

$$B(r,\varphi,\bar{r},\bar{\varphi}) = D(r,\varphi,\bar{r},\bar{\varphi},1), \tag{62}$$

$$\frac{\partial B}{\partial r}(r,\varphi,\bar{r},\bar{\varphi}) = \frac{\partial D}{\partial r}(r,\varphi,\bar{r},\bar{\varphi},1), \tag{63}$$

$$\frac{\partial B}{\partial \bar{r}}(r,\varphi,\bar{r},\bar{\varphi}) = \frac{\partial D}{\partial \bar{r}}(r,\varphi,\bar{r},\bar{\varphi},1), \tag{64}$$

$$\frac{1}{r}\frac{\partial B}{\partial \varphi}(r,\varphi,\bar{r},\bar{\varphi}) = \frac{1}{r}\frac{\partial D}{\partial \varphi}(r,\varphi,\bar{r},\bar{\varphi},1), \tag{65}$$

$$\frac{1}{\bar{r}}\frac{\partial B}{\partial \bar{\varphi}}(r,\varphi,\bar{r},\bar{\varphi}) = \frac{1}{\bar{r}}\frac{\partial D}{\partial \bar{\varphi}}(r,\varphi,\bar{r},\bar{\varphi},1), \tag{66}$$

$$\frac{\partial^2 B}{\partial r\,\partial\bar{r}}(r,\varphi,\bar{r},\bar{\varphi}) = \frac{\partial^2 D}{\partial r\,\partial\bar{r}}(r,\varphi,\bar{r},\bar{\varphi},1), \tag{67}$$

$$\frac{1}{r\bar{r}}\frac{\partial^2 B}{\partial\varphi\,\partial\bar{\varphi}}(r,\varphi,\bar{r},\bar{\varphi}) = \frac{1}{r\bar{r}}\frac{\partial^2 D}{\partial\varphi\,\partial\bar{\varphi}}(r,\varphi,\bar{r},\bar{\varphi},1), \tag{68}$$

$$\frac{1}{r}\frac{\partial^2 B}{\partial\varphi\,\partial\bar{r}}(r,\varphi,\bar{r},\bar{\varphi}) = \frac{1}{r}\frac{\partial^2 D}{\partial\varphi\,\partial\bar{r}}(r,\varphi,\bar{r},\bar{\varphi},1), \tag{69}$$

$$\frac{1}{\bar{r}}\frac{\partial^2 B}{\partial r\,\partial\bar{\varphi}}(r,\varphi,\bar{r},\bar{\varphi}) = \frac{1}{\bar{r}}\frac{\partial^2 D}{\partial r\,\partial\bar{\varphi}}(r,\varphi,\bar{r},\bar{\varphi},1). \tag{70}$$

At the boundary of the domain we prescribe the values $\Phi$ and $\partial\Phi/\partial n$ plotted in Fig. 4. Since the exact solution of the bipotential equation is well known:

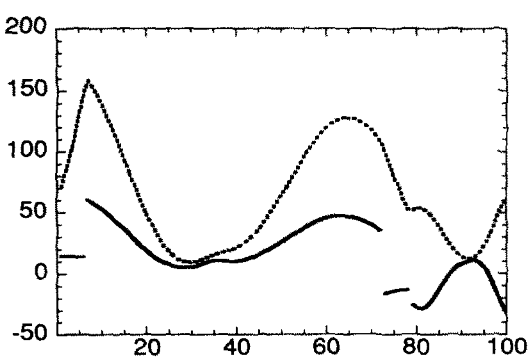


Fig. 4. Boundary values of the function $\Phi$ ····· and of its normal derivative $\partial\Phi/\partial n$ ——— traced out over the straightened boundary. The numbers at the abscissa indicate some of the boundary points (central points of the elements) of Fig. 2.

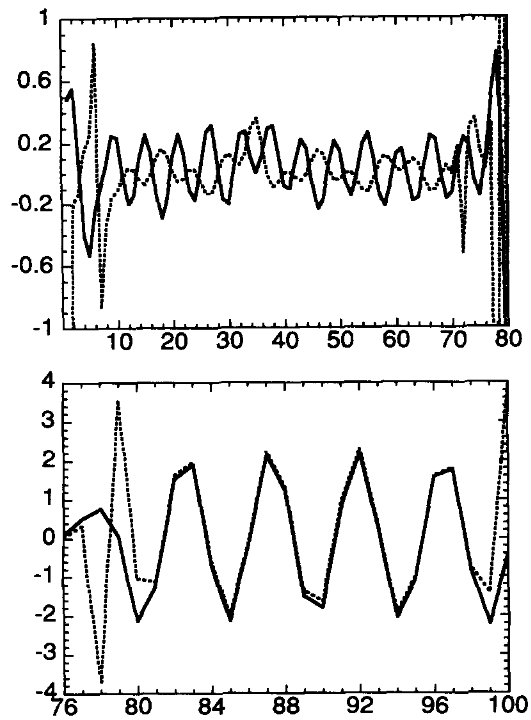


Fig. 5(a, b). Numerical solutions $b(s)$ ····· and $h(s)$ ——— of the integral equation traced out over the straightened boundary. The numbers at the abscissa indicate some of the boundary points (central points of the elements) of Fig. 2. The scales in Figs 5(a) and 5(b) differ from each other! The value $-4.404$ of the function $b(s)$ at point No. 1 lies outside Fig. 5(a).

$$\Phi = [r^2+16-8r\cos(\varphi-\pi)]\ln\left[r^{4/3}+4^{4/3}-2(4r)^{2/3}\cos\frac{\varphi-\pi}{3/2}\right]$$
$$+\left[r^2+4-4r\cos\left(\varphi-\frac{5\pi}{2}\right)\right]\ln\left[r^{2/3}+2^{2/3}-2(2r)^{1/3}\cos\frac{\varphi-5\pi/2}{3}\right] \quad (71)$$

we will be able to check the numerical results.

For discretization of integral equation (39) the boundary is divided into 100 elements of equal lengths. In Fig. 2 the central points of the elements are marked. The unknown functions $b(s)$ and $h(s)$ are prescribed over each individual element as a constant and by collocation at the central points of the elements the integral equation is approximated by a system of algebraic equations. For determining the coefficients of its matrix the integration is carried out with the aid of the simplest quadrature formula, i.e. using one single integration point per element. Only the integrals over the elements containing the field point are evaluated analytically.

The approximate solution is traced out in Fig. 5 over the straightened boundary. Its oscillations are characteristic of the type of integral equations used and not a consequence of having chosen as the kernels a fundamental solution defined on a Riemann surface. The opposed and the coinciding phases of the two curves in Fig. 5 on different parts of the boundary are obviously caused by the coincidence and the noncoincidence of the signs of the derivatives of the boundary values (see Fig. 4). Opposition of phases is related with a considerable reduction of the amplitudes. As is to be expected the solution has peaks at the four corners of the boundary, i.e. at the points 100,1 and 6,7 and 72,73 and 78,79.

From the solution of the integral equation the values of $\Phi$ (e.g. deflection of the plate) are determined at 26 interior points of the domain (see Fig. 5) and compared with the well-known exact values:

| Point No. | Exact value | Error of the numerical value |
|---|---|---|
| 1 | 74.18 | 1.71 |
| 2 | 51.45 | −0.94 |
| 3 | 29.55 | −1.05 |
| 4 | 14.28 | 1.83 |
| 5 | 8.02 | −1.29 |
| 6 | 8.60 | −1.26 |
| 7 | 12.86 | 1.68 |
| 8 | 21.40 | −1.42 |
| 9 | 34.83 | −0.90 |
| 10 | 50.73 | 1.82 |
| 11 | 64.90 | −1.34 |
| 12 | 73.32 | −0.64 |
| 13 | 73.68 | 1.71 |
| 14 | 98.61 | 0.59 |
| 15 | 65.95 | −0.43 |
| 16 | 34.74 | −0.53 |
| 17 | 13.86 | 0.41 |
| 18 | 7.10 | −0.48 |
| 19 | 9.88 | −0.85 |
| 20 | 14.92 | 0.34 |
| 21 | 25.66 | −0.61 |
| 22 | 44.25 | −0.47 |
| 23 | 66.80 | 0.44 |
| 24 | 86.73 | −0.38 |
| 25 | 98.00 | −0.14 |
| 26 | 97.27 | 0.46 |

(72)

The given numerical results have been obtained with the aid of a fundamental solution defined on a Riemann surface with $n = 2$ sheets. Obviously a number of sheets $n > \alpha/(2\pi)$ has to be chosen where $\alpha$ is the aperture angle of the domain, i.e. $n > 7.3333/(2\pi)$ in the case of the considered sector of Fig. 2. Further numerical results show that the condition of the algebraic system gets worse with increasing numbers $n$.

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